

# Gravitational Lensing in Inhomogeneous Universes

Robert M. Wald

*Enrico Fermi Institute and Department of Physics*

*University of Chicago*

*5640 S. Ellis Avenue*

*Chicago, Illinois 60637-1433*

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## Abstract

I describe a new approach (developed in collaboration with D.E. Holz) to calculating the statistical distributions for magnification, shear, and rotation of images of cosmological sources due to gravitational lensing by mass inhomogeneities on galactic and smaller scales. Our approach is somewhat similar to that used in “Swiss cheese” models, but the “cheese” has been completely eliminated, the matter distribution in the “voids” need not be spherically symmetric, the total mass in each void need equal the corresponding Robertson-Walker mass only on average, and we do not impose an “opaque radius” cut-off. In our approach, we integrate the geodesic deviation equation backwards in time until the desired redshift is reached, using a Monte Carlo procedure wherein each photon beam in effect “creates its own universe” as it propagates. Our approach fully takes into account effects of multiple encounters with gravitational lenses and is much easier to apply than “ray shooting” methods.

In this paper, I will briefly summarize a new approach to determining statistical information on the magnification, shear, and rotation of images of cosmological sources due to gravitational lensing by mass inhomogeneities on galactic and smaller scales. In this approach, one may freely specify an

underlying Robertson-Walker cosmological model together with the relevant information on how the matter is distributed in galaxies. This approach was developed in collaboration with Daniel E. Holz. Full details of the method are given in [1] and some results and applications are given in [1], [2], and [3].

The main assumption underlying our approach concerns the spacetime structure of our universe. It is generally believed that the Robertson-Walker models provide an excellent description of the spacetime metric of our universe “on large scales”. However, Robertson-Walker metrics provide an extremely poor approximation to the description of the actual matter density and spacetime curvature on small scales, which, for example, on Earth are a factor of  $10^{30}$  higher than given by a typical Robertson-Walker model. On the other hand, apart from negligibly small regions of spacetime in the immediate proximity of black holes and neutron stars, Newtonian gravity appears to provide an excellent description of all gravitational phenomena on scales much smaller than the Hubble radius. Thus, the spacetime metric of our universe appears to be such that it corresponds to a Robertson-Walker metric on large scales and a Newtonian metric (relative to the local Robertson-Walker rest frame) on small scales.

A spacetime metric with these features is

$$ds^2 = -(1 + 2\phi) d\tau^2 + (1 - 2\phi)a^2(\tau) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right], \quad (1)$$

provided that

$$|\phi| \ll 1. \quad (2)$$

$$|\partial\phi/\partial\tau|^2 \ll a^{-2}h^{ab}D_a\phi D_b\phi, \quad (3)$$

$$(h^{ab}D_a\phi D_b\phi)^2 \ll h^{ac}h^{bd}D_aD_b\phi D_cD_d\phi. \quad (4)$$

and the matter stress-energy tensor,  $T_{ab}$  (*not* including the cosmological constant term) satisfies

$$T_{ab} \approx \rho u_a u_b, \quad (5)$$

where  $u^a$  is a unit vector which points in the  $(\partial/\partial\tau)^a$  direction. In the above equations,  $h_{ab}$  is either the metric of a unit 3-sphere ( $k = 1$ ), a unit 3-hyperboloid ( $k = -1$ ), or flat 3-space ( $k = 0$ )

$$h_{ab} \equiv \frac{1}{1 - kr^2} dr_a dr_b + r^2(d\theta_a d\theta_b + \sin^2\theta d\varphi_a d\varphi_b) \quad (6)$$

and  $D_a$  denotes the spatial derivative operator associated with  $h_{ab}$ . In essence, eq.(2) ensures that the large scale structure of the universe is well described by an *underlying Robertson-Walker model* (i.e., the metric obtained by setting  $\phi = 0$  in eq.(1)). Equation (3) ensures that the metric is locally quasi-static in the rest frame of a Robertson-Walker observer (so that, in particular, negligible gravitational radiation is present) and eq.(4) ensures that the nonlinear contributions of  $\phi$  to the curvature are negligible compared with the linear contributions. These latter two conditions are necessary for Newtonian gravity to be a good approximation locally (i.e., on scales small compared with the Hubble radius) in the Robertson-Walker rest frame. Finally eq.(5) assumes that matter stresses are small compared with energy densities and that the matter does not move at speeds comparable to  $c$  relative to the rest frame of the underlying Robertson-Walker model. It is important to note that eqs.(2)-(5) *do* permit the spatial derivatives of  $\phi$  to locally be very large compared with scales set by the curvature of the underlying Robertson-Walker model.

As found in [1], the metric (1) and eqs.(2)-(4) are compatible with Einstein's equation<sup>1</sup> provided that  $a$  is related to the spatially averaged mass density  $\bar{\rho}$  by the usual Robertson-Walker equations

$$3\ddot{a}/a = \Lambda - 4\pi\bar{\rho} \quad (7)$$

$$3(\dot{a}/a)^2 = \Lambda + 8\pi\bar{\rho} - 3k/a^2, \quad (8)$$

and  $\phi$  satisfies

$$a^{-2}h^{ab}D_aD_b\phi = 4\pi\delta\rho. \quad (9)$$

It then follows [1] that in a local inertial frame associated with any Robertson-Walker observer, the metric takes the form

$$ds^2 = -(1 + 2\Phi - \Lambda R^2/3) dT^2 + (1 - 2\Phi - \Lambda R^2/6)[dX^2 + dY^2 + dZ^2], \quad (10)$$

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<sup>1</sup>More precisely, if eqs.(7)-(9) hold, then all components of Einstein's equation except the "time-space" components will be satisfied up to terms which are negligible compared with the curvature of the underlying Robertson-Walker model or which can be neglected by eqs.(2)-(4). As noted in [1], the "time-space" components of Einstein's equation contains terms which need not be locally small compared with the Robertson-Walker curvature, in which case one must generalize the metric (1) so as to allow for the presence of nonvanishing "time-space" components of the metric. However, these components will be small compared with  $\phi$  and will have completely negligible effects on lensing.

where  $R^2 = X^2 + Y^2 + Z^2$  and terms of order  $(R/R_H)^3$  and higher have been dropped, where  $R_H \equiv a/\dot{a}$ . Here

$$\Phi \equiv \phi + 2\pi R^2 \bar{\rho}/3, \quad (11)$$

satisfies the ordinary Poisson equation

$$\nabla^2 \Phi = 4\pi \rho. \quad (12)$$

Thus, eq.(10) describes a Newtonian perturbation of Minkowski spacetime. Consequently our cosmological model is one which corresponds closely to a Robertson-Walker model insofar as the causal structure of the spacetime and the large scale Hubble flow of the matter are concerned. However, the local distribution of matter may be highly inhomogeneous. Nevertheless, on scales small compared with those set by the underlying Robertson-Walker model, Newtonian gravity holds to a very good approximation. Apart from negligibly small regions of spacetime which contain black holes or other strong field objects, I see no reason to doubt that our universe is accurately described by this model.

It should be noted that the metric (1) has many special features, such as the presence of an irrotational, shear free congruence—namely, the preferred observers of the underlying Robertson-Walker metric—with respect to which the magnetic part of the Weyl tensor vanishes. Ellis and van Elst [4] recently have found that, apart from the Robertson-Walker models, there exist few, if any, exact solutions of Einstein's with these special features. However, there is no reason to expect these special features to survive when higher order corrections to the metric are taken into account; indeed the corrections mentioned above in footnote 1 already violate these features. The non-existence of exact solutions with these features is entirely consistent with the metric (1) providing an excellent approximation to the actual spacetime metric and curvature of our universe.

Consider, now, the propagation of an infinitesimal beam of photons. All gravitational focusing and shearing effects are described by the geodesic deviation equation

$$\frac{d^2 \eta^a}{d\lambda^2} = -R_{bcd}{}^a k^b k^d \eta^c, \quad (13)$$

where  $k^a$  is the tangent to the null geodesic,  $\lambda$  is the corresponding affine parameter, and  $\eta^a$  is the deviation vector to an infinitesimally nearby null geodesic in the beam. In a Robertson-Walker model only Ricci curvature

is present, and it causes the beam to continually undergo a small degree of focusing. On the other hand, in the cosmological model of Eq. (1) in the case where the matter is highly clumped, the Ricci tensor vanishes along the geodesic, except for rare instances when the photon propagates through a clump of matter. On these rare occasions, the Ricci curvature briefly becomes extremely large compared with that of the underlying Robertson-Walker model. The Weyl curvature also will be small except in similarly rare instances of propagation through (or very near) a sufficiently dense clump of matter. Thus, the local history of a photon beam propagating in the spacetime of Eq. (1) can differ enormously from the local history of a photon beam propagating in a Robertson-Walker model.

We wish to accurately calculate probability distributions for the net result of the encounters of a photon beam with matter and Weyl curvature in a spacetime of the form (1) under a wide range of choices of underlying Robertson-Walker model and under a wide range of assumptions about the inhomogeneities of the matter distribution. To do so, we make use of the fact that the universe appears to be Newtonian in a neighborhood of size  $\ll R_H$  of any Robertson-Walker observer. Thus, we can calculate a Newtonian gravitational potential associated with the mass distribution in this neighborhood<sup>2</sup>, obtain its corresponding spacetime curvature, and then propagate the photon beam through this neighborhood using eq.(13). When the beam leaves this neighborhood, we may view it as entering a similar Newtonian neighborhood of another Robertson-Walker observer. Since the Robertson-Walker observers are in relative motion, the photon beam must be redshifted to describe it in the new Newtonian frame, and the changes in the density of matter associated with the expansion of the universe must be taken into account. However, apart from these effects of the global cosmology, the propagation of the photon beam can be described as a sequence of Newtonian encounters with the curvature resulting from local mass distributions.

The above description of the propagation of photon beam provides the basic rationale for our method for calculating probability distributions for the amplification, shear, and rotation of images of cosmologically distant sources. We begin by choosing an underlying Robertson-Walker model and specifying how matter is distributed in galaxies (e.g., truncated isothermal balls of a given density and radius). There is no restriction on how the galactic matter is distributed other than the requirement that the average mass density agree

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<sup>2</sup>Justification for considering only the nearby matter was given in section 1.3 of [1].

with that of the underlying Robertson-Walker model. We also may allow changes in the physical structure of galaxies with time, although we have not done so in most of our simulations. Let  $2\mathcal{R}$  denote the (comoving) average distance between galaxies. We perform a “Monte Carlo” propagation of a beam of photons *backward* in time (starting from the present) in the following manner: We imagine that our photon beam enters a ball of radius  $\mathcal{R}$  centered on a galaxy with a random impact parameter. We then integrate Eq. (13) through the ball using the curvature computed from the Newtonian gravitational potential of the galaxy. When the photon beam exits from this ball, we use the underlying Robertson-Walker model to update the frequency of the photon relative to the local rest frame of the matter, and also to update the proper radius corresponding to the comoving scale  $\mathcal{R}$ . Then we choose another random impact parameter for entry of the photon into a new ball and continue the integration of the geodesic deviation equation. We then repeat this process until the photon has reached the desired redshift. By re-doing this sequence of calculations a large number of times, we build up good statistics on what happens to beams of photons on our past light cone. From this we obtain—for any given Robertson-Walker model and specification of the structure of galaxies—good statistical information on the magnification, shear, and rotation of images of (nearly) point sources at any redshift.

Note that in our Monte Carlo procedure, each photon in effect “creates its own cosmological model” during the course of its propagation. Consequently, when multiple imaging occurs, our approach does not provide a means of directly determining which images are associated with the same source. However, in the case of point masses or truncated isothermal balls, if a single lens dominates (as plausibly would be the case in the strong lensing regime), then standard analytic expressions can be used to obtain the relationship between the magnifications of primary and secondary images of the same source.<sup>3</sup> This allows us to statistically associate secondary images

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<sup>3</sup>A *primary image* is one associated with a photon beam which has not undergone a caustic whereas a *secondary image* is one associated with a beam that has undergone a caustic, and, consequently, has entered the interior of the observer’s past. Every source has at least one primary image, corresponding to a photon beam lying on the boundary of the past of the observer. When the matter in galaxies is distributed in a spherically symmetric manner, we have verified numerically [1] that the total area carried by beams associated with primary images very nearly equals the area of the boundary of the past of the corresponding observer in the underlying Robertson-Walker model, at least out to a redshift of 3. This implies that essentially all primary images lie on the boundary of the past of the observer, and, thus, that essentially all sources have only one primary image.

with primary images.

As explained in [1], our method can easily be adapted to take account of sub-galactic structure on arbitrarily small scales, as would be relevant for microlensing.<sup>4</sup> However, in order to fully take into account effects of the clustering of galaxies themselves, we would need to choose our comoving scale  $\mathcal{R}$  to be the scale of clusters of galaxies, and we then would have to model the mass distribution of the cluster. Fortunately, it does not appear necessary that we take the clustering of galaxies into account in order to obtain accurate results for the statistical distributions of the magnification, shear, and rotation of images. To see this, consider first the limit in which Ricci curvature dominates the lensing effects, i.e., the galaxies are larger than their own Einstein radii and the clustering of the galaxies also is not sufficient strong so as to produce significant Weyl curvature. Since, by Einstein’s equation, the Ricci curvature is determined by the matter distribution in a completely local manner, the lensing effects of galaxies should depend only very weakly on their clustering, since clustering should merely produce some correlations in the times of passage of a photon through different galaxies, but these effects should largely “wash out” over cosmological distance scales. On the other hand, consider the opposite extreme where galaxies lie well within their own Einstein radii, and thus can be treated as “point masses”. In this case, we have shown [1] by a combination of analytic and numerical arguments that arbitrary spherical clustering of the galaxies will have at most a tiny effect on the lensing probability distributions for the magnification, shear, and rotation of (nearly) point sources. Although strong clustering of galaxies will create large scale “cluster potentials”, the additional lensing effects produced by these large scale potentials is almost exactly compensated by “screening effects” on the lensing by individual galaxies, and almost no net change occurs in the probability distributions we calculate. (On the other hand, clustering *would* still have an important effect on some lensing quantities we do not calculate, such as total bending angles.) Thus, clustering of galaxies should be of importance for the statistical lensing quantities we calculate only when individual galaxies are larger than their own Einstein radii (and thus individually contribute only Ricci curvature), but the galaxies cluster into structures that produce significant Weyl curvature. In these

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<sup>4</sup>Clumping of matter on small scales is relevant for lensing only if the angular size of the Einstein radius of the lens is larger than the angular size of the source. For the smallest sources of interest—namely quasars and supernovae—this means that only clumping on mass scales larger than  $10^{-3}M_{\odot}$  is relevant.

circumstances the neglect of the clustering of galaxies should underestimate the lensing effects somewhat. However, we do not believe that such circumstances arise frequently enough to have an important influence on the statistical lensing quantities we calculate.

Our approach is ideally suited to accurately calculate the effects of many different, independent lenses on the image of a source. Although, as noted above, in the strong lensing regime it is expected that a single lens normally will dominate (for moderate source redshifts), this need not be the case in the weak lensing regime. In particular, our approach allows one to accurately calculate the probability that a photon beam will encounter negligible curvature when traversing from the source to the observer, and thus that the magnification of the image of the source will be near its “flat spacetime” (i.e., “empty beam”) value. For reasonable matter distributions, this probability is quite high for sources at redshifts  $< 1$ , so the “empty beam” formula for magnification generally provides at least as good an approximation to the peak of the magnification probability distribution as the Robertson-Walker (i.e., “filled beam”) formula [1], [3]. Consequently, the effects of lensing on the analysis of data from type Ia supernovae can be quite significant [3]. Our method also has already been applied to calculate a number of other effects of cosmological interest, specifically, the correlations between magnification due to lensing and the number of massive gas clouds through which the photon beam passes [1], and the probability for multiple imaging of very high redshift sources [2]. Many additional applications of our method are planned for the near future. In particular, it is planned to use our method to determine the source redshift range over which it may be assumed that a single encounter with a lens dominates strong lensing effects, as well as to accurately calculate the cumulative effects of the sub-dominant lenses [5].

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